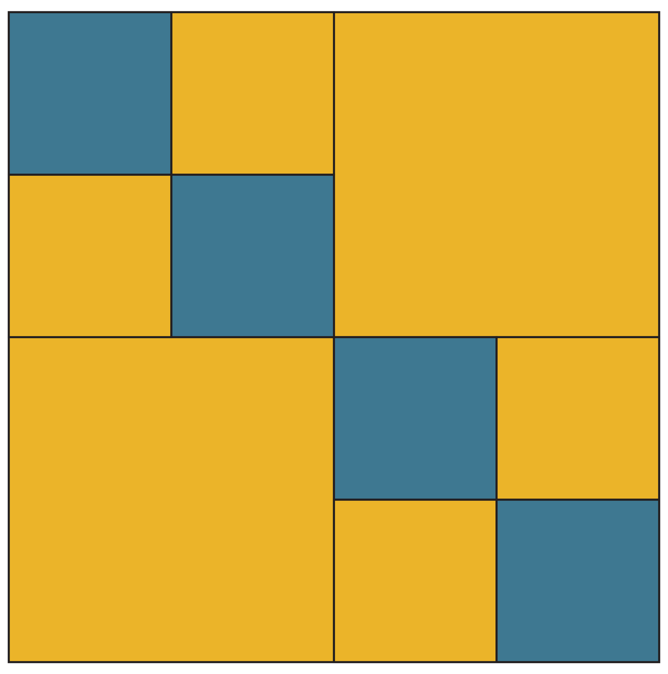
Inertia of HSS matrices using STRUMPACK

Jonas Actor [a], Pieter Ghysels [b], and Xiaoye Li [b]

[a]Computational and Applied Mathematics, Rice University
[b]Scalable Solvers Group, Lawrence Berkeley National Laboratory

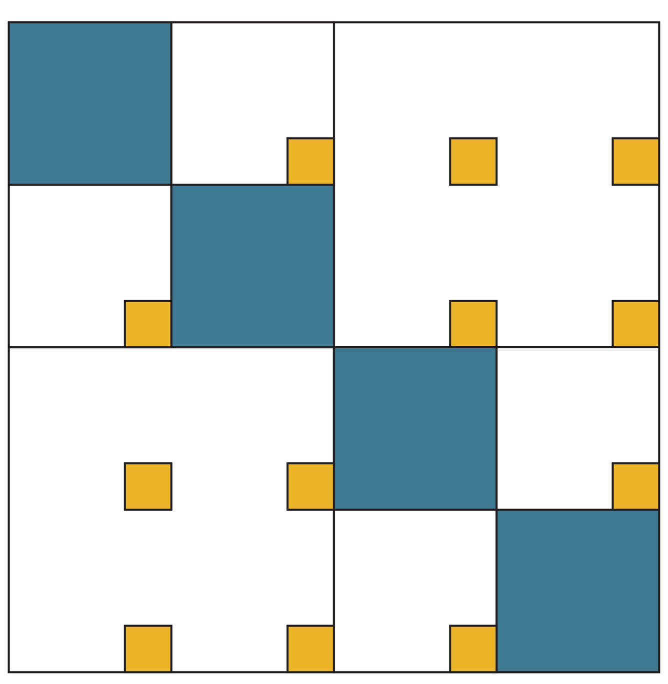


Compression

STRUMPACK compresses a **symmetric indefinite** matrix A (e.g. a KKT matrix) into HSS form.

$$A = \begin{bmatrix} D_1 & U_1^b B_{12} U_2^{b\,T} \\ U_2^b B_{21} U_1^{b\,T} & D_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} D_3 & U_3 B_{34} U_4^T \\ U_4 B_{43} U_3^T & D_4 \end{bmatrix} & \begin{bmatrix} U_3 & \\ & U_4 \end{bmatrix} U_1 B_{12} U_2^T \begin{bmatrix} U_5^T & \\ & U_6^T \end{bmatrix} \\ \begin{bmatrix} U_5 & \\ & U_6 \end{bmatrix} U_2 B_{21} U_1^T \begin{bmatrix} U_3^T & \\ & U_4^T \end{bmatrix} & \begin{bmatrix} D_5 & U_5 B_{56} U_6^T \\ U_6 B_{65} U_5^T & D_6 \end{bmatrix} \end{bmatrix}$$

The **inertia** $\nu(A)$, the count of positive, negative, and zero eigenvalues, is found via bottom-up traversal of the HSS tree.



Introduce Zeros

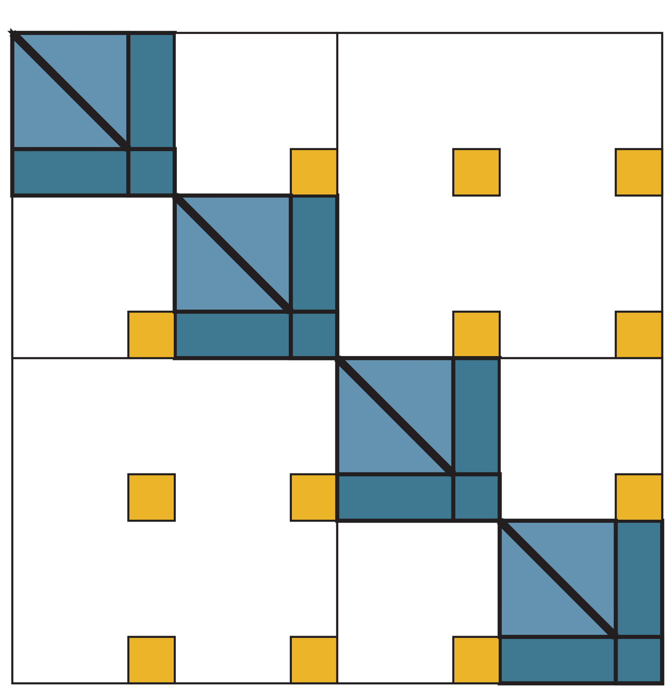
Use Ω_τ matrices from STRUMPACK's HSS factorization to annihilate subblocks, with $\Omega_\tau U_\tau = \begin{bmatrix} 0 \\ I \end{bmatrix}$.

By applying $diag(\Omega_\tau : \tau$ level $\ell)$ to the left and right of A, inertia is preserved.

Partial LDL^T Factorization

Split into four subblocks $\Omega_\tau D_\tau \Omega_\tau^T = \begin{bmatrix} D_{\tau,11} & D_{\tau,12} \\ D_{\tau,21} & D_{\tau,22} \end{bmatrix}$. Take LDL^T factorization of the top left subblock:

$$L_{\tau,11} \widehat{D}_\tau L_{\tau,11}^T = \text{LDL}(D_{\tau,11})$$

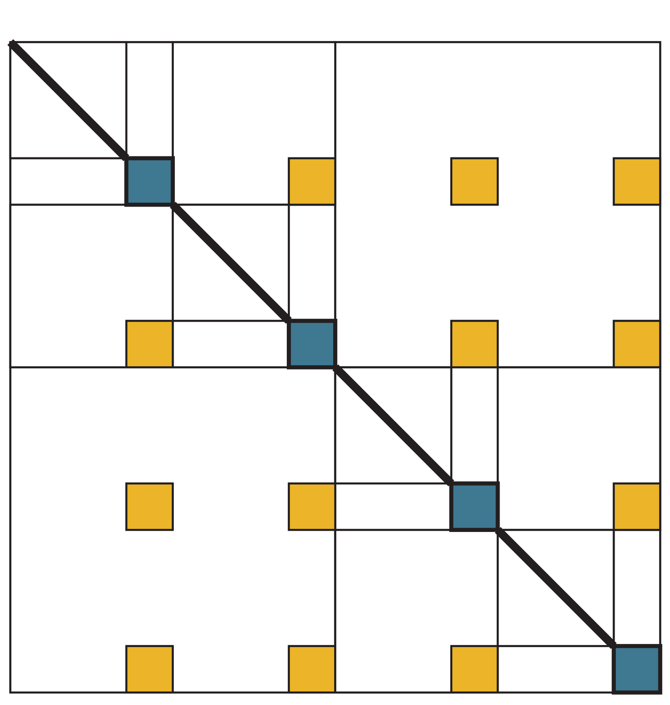


Schur Complement

Form the Schur complement using the LDL^T factors. Use the Schur complement to form an LDL^T factorization of all four subblocks:

$$L_{\tau,21} = D_{\tau,21}(\widehat{D}_\tau L_{\tau,11}^T)^{-1}$$
$$S_\tau = D_{\tau,22} - D_{\tau,21} D_{\tau,11}^{-1} D_{\tau,12}$$
$$\begin{bmatrix} D_{\tau,11} & D_{\tau,12} \\ D_{\tau,21} & D_{\tau,22} \end{bmatrix} = \begin{bmatrix} L_{\tau,11} & \\ L_{\tau,21} & I \end{bmatrix} \begin{bmatrix} \widehat{D}_\tau & \\ & S_\tau \end{bmatrix} \begin{bmatrix} L_{\tau,11} & \\ L_{\tau,21} & I \end{bmatrix}^T$$

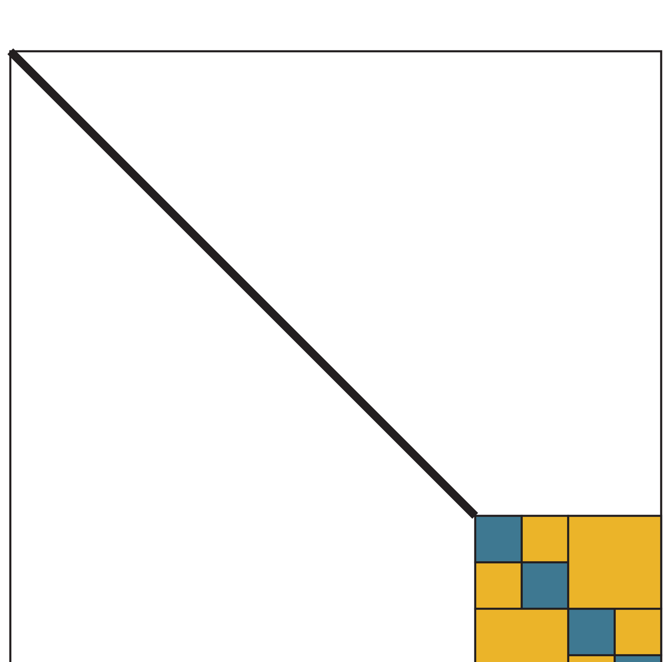
Note inertia is preserved: $\nu(D_\tau) = \nu(\Omega_\tau D_\tau \Omega_\tau^T) = \nu(\widehat{D}_\tau) + \nu(S_\tau)$.



Repeat

Consolidate the remaining factors by merging a factorized node σ_1 with its sibling node σ_2. Assign the consolidated factors to their parent node τ and repeat, thereby moving up the HSS tree.

$$D_\tau \longleftarrow \begin{bmatrix} S_{\sigma_1} & B_{\sigma_1,\sigma_2} \\ B_{\sigma_2,\sigma_1} & S_{\sigma_2} \end{bmatrix}.$$



When at the root, form the whole LDL^T factorization, not just of the upper left subblock. At the end of this recursion,

$$\nu(A) = \sum_{\text{node } \tau \text{ in HSS tree}} \nu(\widehat{D}_\tau).$$

The calculation of $\widehat{D}_\tau$ requires $O(r^3)$ operations. Each level of the HSS tree has for each of $\frac{2^\ell}{r}$ factorizations, and there are $\log_2(N) \approx L_{max}$ levels, providing an overall complexity of $O(r^2 N)$.

low-rank (rank r) blocks **full-rank blocks** **LDL^T factorization** **diagonal blocks**

Selected references: Xi et al., SIMAX 2014; Rouet et al., ACM TOMS 2016.

Contact: `jonasactor@rice.edu`